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The AdS_4/CFT_3 algebraic curve

Nikolay Gromov[abc] **and Pedro Vieira**[bd]

[a]*Service de Physique Théorique, CNRS-URA 2306 C.E.A.-Saclay, F-91191 Gif-sur-Yvette, France*
[b]*Laboratoire de Physique Théorique, Ecole Normale Supérieure et l'Université Paris-VI, Paris, 75231, France*
[c]*St. Petersburg INP, Gatchina, 188 300, St. Petersburg, Russia*
[d]*Departamento de Física e Centro de Física do Porto, Faculdade de Ciências da Universidade do Porto Rua do Campo Alegre, 687, 4169-007 Porto, Portugal*
E-mail: nikgromov@gmail.com, pedrogvieira@gmail.com

Abstract: We present the OSp$(2,2|6)$ symmetric algebraic curve for the AdS_4/CFT_3 duality recently proposed in arXiv:0806.1218. It encodes all classical string solutions at strong t'Hooft coupling and the full two loop spectrum of long single trace gauge invariant operators in the weak coupling regime. This construction can also be used to compute the complete superstring semi-classical spectrum around any classical solution. We exemplify our method on the BMN point-like string.

Keywords: AdS-CFT Correspondence, Integrable Field Theories, Chern-Simons Theories, Bethe Ansatz.

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Contents

1. Introduction and main results

In [1] integrability emerged once again [2] in the study of superconformal gauge theories. In this work Minahan and Zarembo wrote down a set of five Bethe equations yielding the complete 2-loop spectrum of the three dimensional superconformal $\mathrm{SU}(N) \times \mathrm{SU}(N)$ Chern-Simons theory recently proposed by Aharony, Bergman, Jafferis and Maldacena [3] following [4]. This theory was conjectured [3] to be the effective theory for a stack of M2 branes at a Z_k orbifold point. In the large N limit, the gravitational dual becomes M-theory on $AdS_4 \times S^7/Z_k$. For large k and N with

$$\lambda = N/k \equiv 8g^2 \tag{1.1}$$

fixed, the dual theory becomes type IIA superstring theory in $AdS_4 \times CP^3$. For subsequent interesting works see [5, 6, 1, 7, 8]. In [7, 8] the superstring coset sigma model was constructed and shown to be the classically integrable.

In this paper we present the algebraic curve construction for the AdS_4/CFT_3 duality at weak and strong coupling. The curves we present encode the full 2-loop spectrum of long single trace gauge invariant operators in the ABJM Chern Simons theory and the complete classical motion of free type IIA superstring theory in $AdS_4 \times CP^3$. The curve for the AdS_5/CFT_4 Maldacena duality was considered in [9–14].

For the string side we have a supercoset sigma model whose target space is

$$\frac{\mathrm{OSp}(2,2|6)}{\mathrm{SO}(3,1) \times \mathrm{SU}(3) \times \mathrm{U}(1)} \tag{1.2}$$

which has $AdS_4 \times CP^3$ as its bosonic part. The algebraic curve construction allows us to map each classical solution to a corresponding Riemann surface which encodes an infinite set of conserved charges particular to the classical solution in study. The map goes as follows. Given a classical solution we can diagonalize the monodromy matrix

$$\Omega(x) = P \exp \int_\gamma d\sigma J_\sigma(x) \tag{1.3}$$

where $J(x)$ is the flat connection, present for integrable theories and computed for the model in study in [7, 8] and for the $AdS_5 \times S^5$ superstrings in [15], x is an arbitrary complex number called spectral parameter and the integration path is a loop at constant τ. The eigenvalues of the monodromy matrix (as of any generic matrix) live in a Riemann surface whose size is roughly speaking the size of the matrix[1]. In our case, we will see that the logarithms of these eigenvalues (called quasi-momenta) can be organized into a 10-sheeted Riemann surface whose properties are listed below. Due to the flatness of the connection, the quasi-momenta do not depend on τ and thus they encode an infinite set of conserved charges (see the next section for details).

Turning the logic around, the algebraic curve construction allows one to trade the study of the intricate non-linear equations of motion by the construction of Riemann surfaces with prescribed analytical properties, a well developed subject in algebraic geometry.

To illustrate what we mean let us describe the algebraic curve studied in this paper. To find the complete classical spectrum of the theory we should proceed as follows: we should build ten-sheeted Riemann surfaces[2] whose branches, called quasi-momenta, depend on a spectral parameter $x \in \mathbb{C}$ and are denoted by $\{q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}\}$. They are not independent but rather $\{q_1, q_2, q_3, q_4, q_5\} = \{-q_{10}, -q_9, -q_8, -q_7, -q_6\}$. These Riemann surfaces must obey the following analyticity properties:

1. Generically square root cuts may connect different pairs of sheets. When going through each cut the quasi-momenta might gain a multiple integer of 2π,

$$q_i^+ - q_j^- = 2\pi n_{ij} \;\; , \;\; x \in \mathcal{C}_{ij} \tag{1.4}$$

   where the superscript $\pm$ indicates the function is evaluated immediately above/below the square root cut. The set of integers $\{n_{ij}\}$ characterize the several cuts of the Riemann surface, i.e. they are are moduli of the algebraic curve.

2. Each cut is also parametrized by a filling fraction

$$S_{ij} = \frac{g}{\pi i} \oint_{\mathcal{C}_{ij}} dx \left(1 - \frac{1}{x^2}\right) q_i(x) \tag{1.5}$$

   which roughly speaking measures how big the cut is. (From the point of view of the classical solutions these are the action variables.)

[1]For some matrices, such as for example elements of SO$(2N+1)$, some eigenvalues might be trivial.

[2]Actually as will become clear latter the quasi-momenta define an infinite genus curve and to obtain a ten-sheeted Riemann surface we should take for example the derivative of this quasimomenta w.r.t. x.

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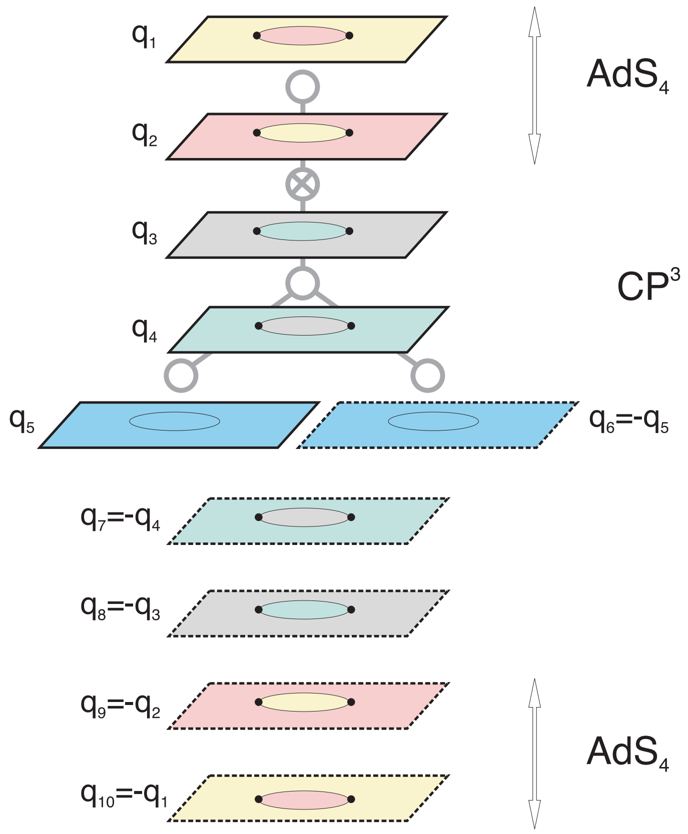

Figure 1: Full $AdS_4 \times CP^3$ algebraic curve in the **10** representation. Poles uniting AdS_4 and CP^3 quasimomenta are fermionic excitations. Regions which are trivially related are painted with the same colour. Poles at $x = \pm 1$ are marked by black filled circles. The OSp$(2, 2|6)$ emerges naturally. Notice that the black dots denoting the poles at $x = \pm 1$ disappear as we jump through the last two Dynkin nodes, whose Dynkin labels are non-zero. This is precisely as expected — Bethe equations are the difference of quasimomenta and therefore this pattern reflects the SU$(N) \times$SU(N) staggered spin chain of Minahan and Zarembo [1] with two momentum carrying nodes.

3. The quasi-momenta must behave as

$$\begin{pmatrix} q_1(x) \\ q_2(x) \\ q_3(x) \\ q_4(x) \\ q_5(x) \end{pmatrix} = - \begin{pmatrix} q_{10}(x) \\ q_9(x) \\ q_8(x) \\ q_7(x) \\ q_6(x) \end{pmatrix} \simeq \frac{1/2}{x \pm 1} \begin{pmatrix} \alpha_\pm \\ \alpha_\pm \\ \alpha_\pm \\ \alpha_\pm \\ 0 \end{pmatrix} \tag{1.6}$$

close to the singular points $x = \pm 1$. The constant $\alpha_\pm$ has no significance from the target space point of view, the only thing we should keep in mind is that the residues must be synchronized (Physically this is a manifestation of the Virasoro constraints imposed on the classical solutions).

4. The curve should possess the inversion symmetry

$$\begin{pmatrix} q_1(1/x) \\ q_2(1/x) \\ q_3(1/x) \\ q_4(1/x) \\ q_5(1/x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2\pi m \\ 2\pi m \\ 0 \end{pmatrix} + \begin{pmatrix} -q_2(x) \\ -q_1(x) \\ -q_4(x) \\ -q_3(x) \\ +q_5(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2\pi m \\ 2\pi m \\ 0 \end{pmatrix} + \begin{pmatrix} +q_9(x) \\ +q_{10}(x) \\ +q_7(x) \\ +q_8(x) \\ -q_6(x) \end{pmatrix} \tag{1.7}$$

with $m \in \mathbb{Z}$.

5. The large x asymptotics of the Riemann surface read[3]

$$\begin{pmatrix} q_1(x) \\ q_2(x) \\ q_3(x) \\ q_4(x) \\ q_5(x) \end{pmatrix} = - \begin{pmatrix} q_{10}(x) \\ q_9(x) \\ q_8(x) \\ q_7(x) \\ q_6(x) \end{pmatrix} \simeq \frac{1}{2gx} \begin{pmatrix} L+E+S \\ L+E-S \\ L-M_r \\ L+M_r-M_u-M_v \\ M_v-M_u \end{pmatrix} . \tag{1.8}$$

If we enumerate all possible curves with these properties then it suffices to evaluate the quasimomenta at large values of the spectral parameter to obtain the energy spectrum of all classical string solutions as a function of the global charges and of the several moduli of the algebraic curve.

Notice that each cut of the algebraic curve is characterized by a discrete label (i, j), corresponding to the two sheets being united, an integer n, the multiple of 2π mentioned above, and a real filling fraction. These three quantities are the analogues of the polarization, mode number and amplitude of the flat space Fourier decomposition of a given classical solution.

This provides us with clear geometrical picture of semi-classical quantization in the context of these algebraic curves [16]. Namely a classical solution will be represented as some algebraic curve with same large cuts uniting several pairs of sheets. Quantum fluctuations correspond to adding small singularities — microscopic cuts or poles — to this Riemann surface [16]. The different choices of sheets to be connected in this way correspond to the different string polarizations we can excite. We will exemplify this procedure on the example of the simplest classical solution — the BMN [17] string studied in the present framework in [6]. This method can be easily generalized to more complicated solutions and to the study of the ground state energy around any classical solution [18].

In the next section we will derive the above mentioned results and explain the construction of the Superstring algebraic curve. In section 3 the gauge theory side of the AdS_4/CFT_3 duality is analyzed: We construct the CFT algebraic curve encoding the spectrum of all long single trace operators at leading order in perturbation theory (2-loops).

[3] The state labeled by (M_u, M_r, M_v) belongs to the SU(4) representation with Dynkin labels $[L-2M_u+M_r, M_u+M_v-2M_r, L-2M_v+M_r]$

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2. String algebraic curve

In this paper we construct the algebraic curve for free type IIA superstrings in AdS_4/CP^3. We could use the flat connection in [7, 8] which, superficially, has the same form as that found by Benna Polchinki and Roiban for the $AdS_5 \times S^5$ strings [15]. Armed with the experience of what happens in the AdS_5/CFT_4 duality [9, 14, 16] we follow a shortcut[4]. We shall use the purely bosonic part of the action to compute the CP^3 and the AdS_4 algebraic curves. They will be coupled solely by the Virasoro constraints. Then, to lift the classical bosonic curve to the complete semi-classical curve for the full super group, we will simply allow the several sheets of the two curves to be connected by further small cuts or poles. Such poles connecting CP^3 sheets AdS_4 sheets correspond to the missing fermionic excitations.

Technically, our treatment is very similar to the one in [11] where the SO(6) bosonic string was studied and the generalization to SO($2n$) was presented. This is not surprising since OSp$(2, 2|6)$ is not very different from SO(10).

2.1 Bosonic flat connection

The bosonic part of the AdS_4/CP^3 type IIA free superstring theory reads

$$S = \sqrt{2\lambda} \int d\sigma d\tau \, (\mathcal{L}_{CP^3} + \mathcal{L}_{AdS^4}) \tag{2.1}$$

where

$$\mathcal{L}_{AdS^4} = -\frac{1}{4} \left(\partial_\mu n \cdot \partial_\mu n - \Lambda \, (n \cdot n - 1)\right) , \tag{2.2}$$

and

$$\mathcal{L}_{CP^3} = (D_\mu z)^\dagger \cdot D_\mu z - \Lambda' \left(z^\dagger \cdot z - 1\right) . \tag{2.3}$$

Here n and z are vectors made out of the embedding coordinates of the anti de-Sitter and the projective space. Thus

$$n = (n_1, \ldots, n_5) \ , \ n \cdot n = n_1^2 + n_2^2 - n_3^2 - n_4^2 - n_5^2 \tag{2.4}$$

with n_i real while

$$z = (z^1, \ldots, z^4) \ , \ z^\dagger \cdot z = |z^1|^2 + \cdots + |z^4|^2 \tag{2.5}$$

where z^I are complex numbers. In what follows whenever the index structure is obvious we omit it. The z^I are also identified up to a phase, $z^I \simeq e^{i\varphi} z^I$. This U(1) gauge symmetry is accounted by the gauge field A_μ appearing in

$$D_\mu z = \partial_\mu z + i A_\mu z \, . \tag{2.6}$$

The equations of motion for the connection yield $z \cdot (D_\mu z)^\dagger - (D_\mu z) \cdot z^\dagger = 0$ while the constrain $\partial_\mu (z^\dagger \cdot z) = 0$ yields $z \cdot (D_\mu z)^\dagger + (D_\mu z) \cdot z^\dagger = 0$ and therefore, on-shell, we have separately

$$(D_\mu z) \cdot z^\dagger = z \cdot \left(D_\mu z^\dagger\right) = 0 \, . \tag{2.7}$$

[4] Of course, if we would proceed as in [14] using the full flat connection in [7, 8] we would find exactly the same results, as can be easily checked.

Analogously, for the n field we have

$$n \cdot (\partial_\mu n) = 0\,. \tag{2.8}$$

Next it is useful to introduce the element

$$h = \left(\begin{array}{c|c} 1 - 2\, z^\dagger \otimes z & \\ \hline & 1 - 2\, n \otimes n \end{array}\right) \;\Leftrightarrow h_{AB} = \left(\begin{array}{c|c} \delta_I^J - 2\, z_I^\dagger z^J & \\ \hline & \delta_{ij} - 2\, n_i n_j \end{array}\right)\,, \tag{2.9}$$

and the connection

$$j = h^{-1} dh \equiv \left(\begin{array}{c|c} j_{\mathrm{AdS}} & \\ \hline & j_{\mathrm{CP}} \end{array}\right)\,. \tag{2.10}$$

It is easy to see that when $z^\dagger \cdot z = n \cdot n = 1$ and (2.7), (2.8) hold we have

$$j_{AB} = 2 \left(\begin{array}{c|c} n_i\,(\partial_\mu n_j) - (\partial_\mu n_i)\, n_j & \\ \hline & z_I^\dagger\,(D_\mu z)^J - (D_\mu z)_I^\dagger\, z^J \end{array}\right) \tag{2.11}$$

and the action can be written using this current as

$$S = -\frac{g}{4} \int d\sigma d\tau\, \mathrm{STr}_{\mathrm{OSp}}\left(j_\mu^2\right)\,. \tag{2.12}$$

where[5]

$$\mathrm{STr}_{\mathrm{OSp}}\left(j^2\right) \equiv -\frac{1}{2}\mathrm{Tr}\left(j_{\mathrm{AdS}}^2\right) + 2\mathrm{Tr}\left(j_{\mathrm{CP}}^2\right) \tag{2.13}$$

Note also that the Virasoro constraint now implies

$$\mathrm{STr}_{\mathrm{OSp}}\left(j_1 \pm j_0\right)^2 = 0\,. \tag{2.14}$$

Furthermore we have the flatness condition

$$dj + j \wedge j = 0\,, \tag{2.15}$$

following from the expression (2.10) of the connection, and the conservation

$$d * j = 0\,, \tag{2.16}$$

which is equivalent to the equations of motion for both the n and the z field. These two equations follow from the flatness condition for the Lax connection

$$J(x) = \frac{j + x * j}{1 - x^2} \tag{2.17}$$

[5]This definition is motivated by the relations between quadratic Casimirs for $\mathrm{SO}(5), Sp(4)$ and $\mathrm{SU}(4), \mathrm{SO}(6)$

$$\mathrm{Tr}_{\mathrm{SO}(5)}(j^2) = 2\mathrm{Tr}_{\mathrm{Sp}(4)}(j^2)\;\;,\;\; \mathrm{Tr}_{\mathrm{SU}(4)}(j^2) = \frac{1}{2}\mathrm{Tr}_{\mathrm{SO}(6)}(j^2)$$

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as can be easily checked by collecting powers of x in $dJ(x)+J(x)\wedge J(x)$. The new variable x appearing in (2.17) is a completely arbitrary complex number called spectral parameter. Using this flat connection we can build the monodromy matrix

$$\Omega(x) = P\exp\int d\sigma J_\sigma(x) \tag{2.18}$$

where we integrate over a constant τ worldsheet loop. At this point integrability comes into stage. The connection being flat, the eigenvalues of the monodromy matrix are independent of τ. Since, moreover, they depend on a generic complex number x they define an infinite set of conserved charges. For example, each coefficient in the taylor expansion of the eigenvalues around a particular point x^* is a conserved charge. The existence of this large number of conserved charges render the sigma model (at least classically) integrable[6].

We want to study the algebraic curve construction for this integrable model. This will map each classical string solution to a Riemann surface with precise analytical properties. The study of classical solutions in $AdS_4\times CP^3$ can then be reduced to the problem of making a catalogue of all Riemann surfaces compatible with prescribed analytical properties.

2.2 The $AdS_4 \times CP^3$ algebraic curve

In this section we study the eigenvalues of the monodromy matrix (2.18). We will first consider purely bosonic solutions and work out the full supercurve in the next section. From (2.17) and (2.10) we see that the flat connection $J(x)$ is explicitly block diagonal and thus the eigenvalues of the monodromy matrix will split into two groups: The eigenvalues coming from the CP^3 part

$$\{e^{i\tilde{p}_1},\dots,e^{i\tilde{p}_4}\}\,, \tag{2.19}$$

and those coming from the diagonalization of the AdS block,

$$\{e^{i\hat{p}_1},\dots,e^{i\hat{p}_4},1\}\,. \tag{2.20}$$

Moreover, from the fact that each of the blocks in (2.11) is manifestly traceless we get

$$\tilde{p}_1(x)+\cdots+\tilde{p}_4(x)=0\,, \tag{2.21}$$

and

$$\hat{p}_3(x)+\hat{p}_2(x)=0=\hat{p}_4(x)+\hat{p}_1(x)\,. \tag{2.22}$$

To find the eigenvalues of the monodromy matrix we solve a polynomial characteristic equation. This defines a algebraic curve for the eigenvalues λ. Thus, the eigenvalues can the thought of as different branches of the same Riemann surfaces with square root cuts uniting the several sheets. For example when crossing a cut $\mathcal{C}$ shared by the eigenvalues $e^{i\hat{p}_1}$ and $e^{i\hat{p}_2}$ we simply change Riemann sheet,

$$\left(e^{i\hat{p}_2}\right)^+ - \left(e^{i\hat{p}_1}\right)^- = 0\ ,\ \ x\in\mathcal{C} \tag{2.23}$$

[6]Quantum integrability is of course a more subtle matter. In particular, a quantum anomaly for the bosonic CP^3 sigma-model is known to exist and break down integrability for this theory. In any case, we will consider the semi-classical quantization of the full super-string where supersymmetry will most likely ensure integrability at the quantum level.

where the superscript $\pm$ indicates the function is evaluated immediately above/below the cut. The quasi-momenta, on the other hand, are the logarithms of the eigenvalues. Thus when crossing the very same cut they will in general also gain an integer multiple of 2π,

$$\hat{p}_2^+ - \hat{p}_1^- = 2\pi n \ , \ \ x \in \mathcal{C} \, . \tag{2.24}$$

Generically we can have several cuts uniting different pairs of sheets and

$$p_i^+ - p_j^- = 2\pi n \ , \ \ x \in \mathcal{C}_{ij} \, , \tag{2.25}$$

on each cut connecting two quasimomenta p_i and p_j. To parametrize each cut we also introduce the usual filling fractions

$$\hat{S}_{ij} = \frac{g}{2\pi i} \oint_{\mathcal{C}_{ij}} dx \left(1 - \frac{1}{x^2}\right) \hat{p}_i(x) \ \ , \ \ \tilde{S}_{ij} = \frac{g}{\pi i} \oint_{\mathcal{C}_{ij}} dx \left(1 - \frac{1}{x^2}\right) \tilde{p}_i(x) \tag{2.26}$$

for the cuts uniting p_i and p_j. Each cut of the algebraic curve is characterized by a discrete label (i, j), corresponding to the two sheets being united, an integer n, the multiple of 2π mention above, and a real filling fraction. These three quantities are the analogues of the polarization, mode number and amplitude of the flat space Fourier decomposition of a given classical solution.

The study of the analytical properties of the quasi-momenta follows closely the analysis done in the context of the AdS_5/CFT_4 duality in [14]. Let us enumerate all these properties and then explain their origin.

For large values of the spectral parameter, the quasimomenta behave as

$$\begin{aligned}
(\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4) &\simeq \frac{1}{gx} \left(L + E, S, -S, -L - E\right) \, , \\
(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4) &\simeq \frac{1}{2gx} \left(L - M_u, M_u - M_r, M_r - M_v, -L + M_v\right) \, ,
\end{aligned} \tag{2.27}$$

for a state belonging to the SU(4) representation with Dynkin labels $[L - 2M_u + M_r, M_u + M_v - 2M_r, L - 2M_v + M_r]$ (which should be positive).

There are two simple poles at $x = \pm 1$ which are synchronized between the AdS_4 and the CP^3 quasi-momenta,

$$(\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4; \tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4) \simeq \frac{1}{x \pm 1} \left(\alpha_\pm, 0, 0, -\alpha_\pm; \frac{\alpha_\pm}{2}, 0, 0, -\frac{\alpha_\pm}{2}\right) \, , \tag{2.28}$$

and finally the algebraic surface exhibits a $x \to 1/x$ inversion symmetry under which

$$\begin{array}{ll}
\hat{p}_1(1/x) = -\hat{p}_1(x) & \tilde{p}_1(1/x) = \tilde{p}_4(x) + 2\pi m \\
\hat{p}_2(1/x) = +\hat{p}_2(x) & \tilde{p}_2(1/x) = \tilde{p}_2(x) \\
\hat{p}_3(1/x) = +\hat{p}_3(x) \quad , & \tilde{p}_3(1/x) = \tilde{p}_3(x) \\
\hat{p}_4(1/x) = -\hat{p}_4(x) & \tilde{p}_4(1/x) = \tilde{p}_1(x) - 2\pi m
\end{array} \, . \tag{2.29}$$

where m is an integer which depends on the classical solution to which these quasi-momenta are associated.

Let us now briefly explain the origin of these analytical properties. The fact that the quasi-momenta encode the global charges of the classical solutions at the $x \to \infty$ asymptotics follows from the large x behavior of the monodromy matrix,

$$\Omega(x) \simeq 1 + \frac{1}{x} \int d\sigma j_\tau \,. \tag{2.30}$$

From the form of the flat connection we see that in general the quasi-momenta can have simple poles at $x = \pm 1$. The reason why only four of them — two in CP^3 and two in AdS_4 — have non-vanishing residues follows from the very particular form of the flat connection. For example for $x \simeq 1$ we have $J(x) \propto j_+$ and thus

$$j_+ \cdot v = 0 \tag{2.31}$$

if v is orthogonal to both $z_I^\dagger$, $D_+ z_I^\dagger$, n_i and $\partial_+ n_i$. Following the arguments in [11], this can be shown to imply that only two CP^3 and two AdS_4 quasimomenta have poles. Since (2.21) and (2.22) we immediately see that the residues at these poles must be symmetric. Moreover the Virasoro constraints $\mathrm{STr}(j_\mu^2) = 0$ synchronizes the poles in the anti de-Sitter and projective space as in (2.28) (exactly as in [13]).

Next we notice that $h^{-1} = h$. This has important consequences for the algebraic curve. It implies that

$$\Omega(x) = h^{-1}(2\pi)\Omega(1/x)h(0) \tag{2.32}$$

and therefore the eigenvalues of the monodromy matrix associated to some closed string classical solution are at most exchanged between themselves under the inversion map $x \to 1/x$. The precise way in which the quasi-momenta are exchanged is in general a subtle business [11, 13] and (2.29) is not a trivial relation. For example, a priori from (2.32) it seems that we could not infer that $\tilde{p}_2(1/x)$ is not exchanged with $\tilde{p}_3(x)$. The reason why this can not happen and the inversion symmetry we postulated is correct is the following: There are solutions with $z_1, z_2, z_3 \neq 0$ but $z_4 = 0$. For those solutions the last line and column of the current $J(x)$ is made out of zeros. Thus $\Omega(x)$ will have one eigenvalue exactly equal to 1 and therefore one of the quasimomenta is strictly zero. Since $\tilde{p}_1$ and $\tilde{p}_4$ have poles the vanishing quasimomenta must be either $\tilde{p}_2$ or $\tilde{p}_3$. But then, if for example $\tilde{p}_3 = 0$ while the other three quasimomenta are nontrivial then, clearly, $p_2(1/x) \neq p_3(x)$! In the same way we can justify the remaining relations in (2.29)[7].

2.3 Full algebraic supercurve

In this section we generalize the classical bosonic algebraic curve described in the previous section to the semi-classical and supersymmetric OSp(2, 2|6) algebraic curve. The smallest representation of this symmetry group — which behaves in many aspects as SO(10) — is **10**

[7]As we mentioned in the beginning we could also have constructed the curve using the flat connection in [7, 8]. The $x \to 1/x$ symmetry we just discussed appears in these works as a consequence of the $\mathbb{Z}_4$ grading of the superalgebra (see equation at the end of section 4.1 in [8]). In the context of the $AdS_5 \times CFT_4$ correspondence — see [13] — this was also the case. Had we used the flat connections in these works and we would have found the same inversion symmetry properties.

dimensional so we should find a nice linear combination of the quasimomenta in the previous section yielding **10** functions describing a single algebraic curve with manifest OSp(2, 2|6) symmetry. Then, to include fermions, we simply allow for extra poles between the AdS_4 and the CP^3 quasimomenta! A proper linear combination is the following reorganization of the quasimomenta into a set of ten functions

$$\{q_1, q_2, q_3, q_4, q_5\} = \left\{\frac{\hat{p}_1 + \hat{p}_2}{2}, \frac{\hat{p}_1 - \hat{p}_2}{2}, \tilde{p}_1 + \tilde{p}_2, \tilde{p}_1 + \tilde{p}_3, \tilde{p}_1 + \tilde{p}_4\right\}, \tag{2.33}$$

and

$$\{q_6, q_7, q_8, q_9, q_{10}\} = -\{q_1, q_2, q_3, q_4, q_5\}. \tag{2.34}$$

These ten functions can be thought of as the several sheets of a single function taking values in a ten-sheeted Riemann surface as represented in figure 1. Notice that they organize in a nice explicitly OSp(2, 2|6) symmetric way. From the properties derived in the previous section for $p_i(x)$ the relations listed in the introduction for $q_i(x)$ follow.

Next, to understand the quasi-classical quantization of any classical solution we add extra pole singularities to the different pairs of sheets of the algebraic curve associated with the solution we want to quantize. The several pairs of sheets to be connected in figure 1 correspond to the different physical polarizations for the quantum fluctuations. Thus, we are in need of a map between the several possible excitations of the string Hilbert space (or of the dual gauge theory) and the several pairs or Riemann surfaces. This map is provided by figure 2 where we listed all $16 = 8 + 8$ physical excitations. The fluctuations were identified by the corresponding excitations of the OSp(2, 2|6) Dynkin diagram, as in [1]. Since the asymptotics of the curve are also be related to the Dynkin labels of a given state this suffices to identify which pairs of sheets are connected for each quantum fluctuation. See [13, 16] for similar analysis in the context of the AdS_5/CFT_4 duality.

In the next section we will explicitly apply figure 2 to the semi-classical quantization of the BMN string.

2.4 BMN string

The BMN point-like string has $z_1 = \frac{1}{\sqrt{2}}e^{i\omega\tau/2}$, $z_2 = \frac{1}{\sqrt{2}}e^{-i\omega\tau/2}$ and $n_1 + in_2 = e^{i\omega\tau}$. Computing the charges of this solution we find

$$L = 4\pi g\,\omega = \pi\sqrt{2\lambda}\,\omega \quad , \quad E = 0 \tag{2.35}$$

to check that one can use that the AdS_5 time is given by $-i\log(n_1 + in_2)$. To compare the results we will find below with those in [6] we notice that

$$\frac{n^2}{\omega^2} = \frac{2\pi^2\lambda\, n^2}{L^2}. \tag{2.36}$$

We now plug the embedding coordinates into (2.11) and compute the path ordered exponential in (2.18). Since the string is point-like there is no σ dependence and this computation

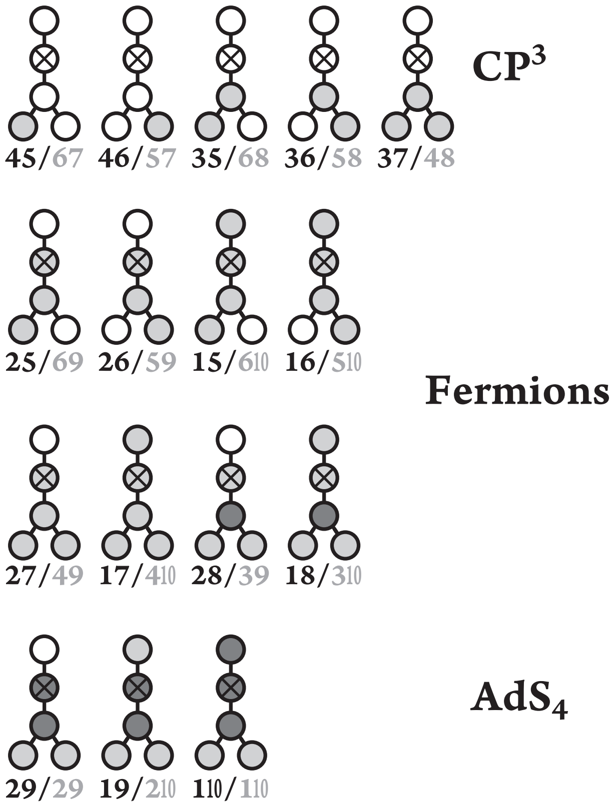

Figure 2: The several states in the Hilbert space can be constructed in the usual oscillator representation. There is one oscillator per Dynkin node of the OSp$(2,2|6)$ super Dynkin diagram. A light (dark) gray shaded node corresponds to an oscillator excited once (twice). From the Chern-Simons Bethe ansatz point of view, the number of times each oscillator is excited is the same as the number of Bethe roots of the corresponding type. Thus, for example, in the notation of [1], the last fermionic excitation corresponds to a bound state of one root of each type u, v, w, s and two Bethe roots r. From the string point of view fluctuations correspond to poles uniting the several sheets of the algebraic curve. Close to each fluctuation we represented some numbers which indicate which quasi-momenta are united. For example we have 45/67 for the first fluctuation which means q_4 and q_5 share a pole. Since (2.34) automatically q_6 and q_7 also share a pole.

is trivial. We can then compute the eigenvalues of the monodromy matrix (2.18) and from them we find the **10** q_i's using (2.33) and (2.34). We obtain

$$q_{1,\ldots,4} = -q_{6,\ldots,10} = \frac{2\pi\omega x}{x^2-1}\,, \tag{2.37}$$

and

$$q_{5,6} = 0\,. \tag{2.38}$$

The BMN string is the simplest possible algebraic curve. It is in fact the *vacuum curve*, all sheets are empty except for the two single poles at $x = \pm 1$.

We can now exemplify the computation of the quasi-classical spectrum in the algebraic curve language and reproduce the recent results of [6]. The 16 physical excitations are represented in figure 2. Notice that the first four CP^3 fluctuations and the last four

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fermionic fluctuations corresponds to poles shared by a quasimomenta in the list (2.37) with one of the two quasimomenta in (2.38). The position of these fluctuations is given by [16]

$$q_i(x_n) - q_j(x_n) = 2\pi n\,. \tag{2.39}$$

The integer n is the generalization of the Fourier mode in flat space — it is meaningful around any classical solution, no matter how non-linear and non-trivial this solution might be. For the fluctuations we are discussing this equation reads

$$\frac{2\pi\omega x_n}{x_n^2-1} = 2\pi n\,, \tag{2.40}$$

and we should pick the solution in the physical region $|x|>1$. On the other hand all the remaining eight fluctuations connect two quasimomenta in (2.37). Therefore, from (2.39), we will find that the position of these fluctuations is fixed by

$$\frac{2\pi\omega x_n}{x_n^2-1} = \pi n\,. \tag{2.41}$$

So the position of half of the fluctuations is the same as the position of the other half with doubled mode number. This already points towards the structure of the fluctuation energies observed in [6]. We also recall that a fluctuation pole at position y should have a residue [16]

$$\alpha(y) = \frac{1}{2g}\frac{y^2}{y^2-1}\,. \tag{2.42}$$

We will now consider separately the CP^3, AdS_4 and Fermionic excitations to understand how to compute the fluctuation energies around a classical solution in the algebraic curve formalism. The computations are conceptually as in [16] so we will simply present the results for the perturbed quasi-momenta with very few explanations.

A technical detail: When computing the fluctuation spectrum we will add always a fluctuation with mode number n and another with mode number $-n$ to keep the string total world-sheet momentum zero in the process[8].

2.4.1 CP^3 excitations

There are two types of fluctuations in CP^3: The first four in figure 2 and the fifth one. The former corresponds to a pole connecting a quasi-momenta in (2.37) with an empty one in (2.38) whereas the latter corresponds to a pole shared by q_3 and q_7, both in (2.37). Let us consider one of the fluctuations of the first type, say the first one in figure 2. For this

[8]We could alternatively excite all polarizations at the same time while obeying the level matching condition

$$\sum_{ij,n} n N_n^{ij} = 0\,, \tag{2.43}$$

with N_n^{ij} being the number of fluctuations with polarization (i,j) and mode number n. This would lead to the same results but would cluster our expressions. Thus we will chose to add always a single pair of fluctuations with mode numbers $\pm n$ at a time.

fluctuation

$$\delta q_3 = -\delta q_8 = +\sum_{\pm} \frac{\alpha(1/x)}{1/x - x_{\pm n}} \tag{2.44}$$

$$\delta q_4 = -\delta q_7 = -\sum_{\pm} \frac{\alpha(x)}{x - x_{\pm n}} \tag{2.45}$$

$$\delta q_5 = -\delta q_6 = +\sum_{\pm} \frac{\alpha(x)}{x - x_{\pm n}} + \sum_{\pm} \frac{\alpha(1/x)}{1/x - x_{\pm n}} \tag{2.46}$$

$$\delta q_{1,2} = -\delta q_{9,10} = +\frac{\alpha(x)\,\delta E}{x}\,. \tag{2.47}$$

so that from the synchronization of poles at $x = \pm 1$ we find

$$\delta E = \sum_{\pm} \frac{1}{x_{\pm n}^2 - 1} = \sum_{\pm n} \sqrt{\frac{1}{4} + \frac{n^2}{\omega^2}} - \frac{1}{2}\,. \tag{2.48}$$

The fifth fluctuation in CP^3 connects q_3 and q_7 (and therefore automatically at $q_4 = -q_7$ and $q_8 = -q_3$). Hence we have

$$\delta q_4 = -\delta q_7 = -\sum_{\pm} \frac{\alpha(x)}{x - x_{\pm n}} + \sum_{\pm} \frac{\alpha(1/x)}{1/x - x_{\pm n}} \tag{2.49}$$

$$\delta q_3 = -\delta q_8 = -\sum_{\pm} \frac{\alpha(x)}{x - x_{\pm n}} + \sum_{\pm} \frac{\alpha(1/x)}{1/x - x_{\pm n}} \tag{2.50}$$

$$\delta q_{1,2} = -\delta q_{9,10} = +\frac{\alpha(x)\,\delta E}{x}\,. \tag{2.51}$$

and we find in this case

$$\delta E = \sum_{\pm} \frac{2}{x_{\pm n}^2 - 1} = \sum_{\pm n} \sqrt{1 + \frac{n^2}{\omega^2}} - 1\,. \tag{2.52}$$

These are precisely the results of [6].

2.4.2 AdS^4 excitations

Here we must be careful. The first and third AdS^4 fluctuations have two excitations in the last Dynkin node — see figure 2. This means that for those we should double the residue (2.42). Let us consider the last one for illustration (for the first one we get the same result of course). We have

$$\delta q_1 = -\delta q_{10} = +\sum_{\pm} \frac{2\alpha(x_n)}{x - x_{\pm n}} \tag{2.53}$$

$$\delta q_2 = -\delta q_9 = -\sum_{\pm} \frac{2\alpha(x_n)}{1/x - x_{\pm n}} \tag{2.54}$$

and thus, from the large x asymptotics,

$$\delta E = \sum_{n=\pm} \frac{x_n^2 + 1}{x_n^2 - 1} = \sum_{\pm n} \sqrt{1 + \frac{n^2}{\omega^2}}\,. \tag{2.55}$$

As for the middle fluctuation in figure 2, we have

$$\delta q_1 = -\delta q_{10} = +\sum_{\pm} \frac{\alpha(x_n)}{x - x_{\pm n}} - \sum_{\pm} \frac{\alpha(x_n)}{1/x - x_{\pm n}} \tag{2.56}$$

$$\delta q_2 = -\delta q_9 = +\sum_{\pm} \frac{\alpha(x_n)}{x - x_{\pm n}} - \sum_{\pm} \frac{\alpha(x_n)}{1/x - x_{\pm n}} \tag{2.57}$$

yielding

$$\delta E = \sum_{n=\pm} \frac{x_n^2 + 1}{x_n^2 - 1} = \sum_{\pm n} \sqrt{1 + \frac{n^2}{\omega^2}} \tag{2.58}$$

which is again the same result as found in [6].

2.4.3 Fermionic excitations

As was the case for the CP^3 excitations, here we also have two types of fluctuations, corresponding to the first and second lines in figure 2. We start by considering a representative of the first line. For example let us focus on a pole from q_1 to q_5 (and thus automatically also from q_6 to q_{10}). We have

$$\delta q_1 = -\delta q_{10} = +\sum_{\pm} \frac{\alpha(x_n)}{x - x_{\pm n}} \tag{2.59}$$

$$\delta q_5 = -\delta q_6 = -\sum_{\pm} \frac{\alpha(x_n)}{x - x_{\pm n}} - \sum_{\pm} \frac{\alpha(x_n)}{1/x - x_{\pm n}} \tag{2.60}$$

$$\delta q_2 = -\delta q_9 = -\sum_{\pm} \frac{\alpha(x_n)}{1/x - x_{\pm n}} \tag{2.61}$$

giving

$$\delta E = \sum_{n=\pm} \frac{x_n^2 + 1}{2(x_n^2 - 1)} = \sum_{\pm n} \sqrt{\frac{1}{4} + \frac{n^2}{\omega^2}} \tag{2.62}$$

For a fluctuation in the second line, say the last one, we have

$$\delta q_1 = -\delta q_{10} = -\sum_{\pm} \frac{\alpha(x_n)}{x - x_{\pm n}} - \frac{Ax}{2g(x^2 - 1)} \tag{2.63}$$

$$\delta q_2 = -\delta q_9 = +\sum_{\pm} \frac{\alpha(x_n)}{1/x - x_{\pm n}} - \frac{Ax}{2g(x^2 - 1)} \tag{2.64}$$

$$\delta q_3 = -\delta q_8 = +\sum_{\pm} \frac{\alpha(x)}{x - x_{\pm n}} \tag{2.65}$$

$$\delta q_4 = -\delta q_7 = -\sum_{\pm} \frac{\alpha(1/x)}{1/x - x_{\pm n}} \tag{2.66}$$

so that pole synchronization gives

$$A = \sum_{\pm} \frac{1}{x_{\pm n}^2 - 1} \tag{2.67}$$

and then from the large x asymptotics we read the energy shift

$$\delta E = \sum_{\pm n} \frac{x_n^2+3}{2(x_n^2-1)} = \sum_{\pm n} \sqrt{1+\frac{n^2}{\omega^2}} - \frac{1}{2} \tag{2.68}$$

This completes the computation of the spectrum of the superstring around the BMN classical solution. All frequencies coincide with those found in [6].

3. Chern-Simons curve

In this section we construct the CS algebraic curve encoding the full 2-loop spectrum of long single trace gauge invariant operators in the ABJM Chern Simons theory. In the scaling limit where the Bethe roots scale with the number of spin chain sites, the two loop Bethe equations in [1] can be recast as [9, 12, 14]

$$\frac{1}{z} + 2\pi n_u = 2\not{G}_u - G_r \tag{3.1}$$

$$\frac{1}{z} + 2\pi n_v = 2\not{G}_v - G_r \tag{3.2}$$

$$2\pi n_r = 2\not{G}_r - G_v - G_u - G_w \tag{3.3}$$

$$2\pi n_w = G_r - G_s \tag{3.4}$$

$$2\pi n_w = 2\not{G}_w - G_s \tag{3.5}$$

In these five equations z belongs to the several disjoint supports where the Bethe roots u, v, r, s, w condense, respectively. As usual

$$G_u = \sum_{j=1}^{M_u} \frac{1}{Lz-u_j} \;\; , \;\; G_v = \sum_{j=1}^{M_v} \frac{1}{Lz-v_j} \;\; , \; \dots \tag{3.6}$$

and the slash means the average of function above and below the cut resulting from the condensation of the Bethe roots. In this limit the spin chain can be described by a (super symmetric) Landau-Lifshitz model and the corresponding algebraic curve can be compared with the curve described in the previous section. Indeed all the 5 nested Bethe equations nested can be turned into the statement that the quasimomenta

$$\begin{aligned}
q_1 = -q_{10} &= \frac{1}{z} - G_w \\
q_2 = -q_9 &= \frac{1}{z} + G_w - G_s \\
q_3 = -q_8 &= \frac{1}{z} \qquad\quad - G_s + G_r \\
q_4 = -q_7 &= \frac{1}{z} \qquad\qquad\qquad - G_r + G_u + G_v \\
q_5 = -q_6 &= \qquad\qquad\qquad\qquad\qquad\; - G_u + G_v
\end{aligned} \tag{3.7}$$

form a ten-sheeted Riemann surface. The several properties of these quasimomenta follow trivially from the definition of the quasimomenta, see [9 – 14]. This curve can be depicted as in figure 1 provided we shrink the unit circle into the origin. The energy of the YM solutions is then given by

$$E = \sum_{i=1}^{M_u} \frac{\lambda^2}{u_i^2} + \sum_{i=1}^{M_v} \frac{\lambda^2}{v_i^2} \,. \tag{3.8}$$

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